# Chapter 3

# Hyperbolic Position Location Systems

## 3.1 Introduction

This chapter introduces the general models for the position location problem and the techniques involved in the hyperbolic position location method. Hyperbolic position location (PL) estimation is accomplished in two stages. The first stage involves estimation of the time difference of arrival (TDOA) between receivers through the use of time delay estimation techniques. The estimated TDOA's are then transformed into range difference measurements between base stations, resulting in a set of nonlinear hyperbolic range difference equations. The second stage utilizes efficient algorithms to produce an unambiguous solution to these nonlinear hyperbolic equations. The solution produced by these algorithms result in the estimated position location of the source. The following sections introduce the techniques and algorithms used to perform hyperbolic position location of a mobile user.

## 3.2 TDOA Estimation Techniques

The time difference of arrival (TDOA) of a signal can be estimated by two general methods: subtracting TOA measurements from two base stations to produce a relative TDOA, or through the use of cross-correlation techniques, in which the received signal at one base station is correlated with the received signal at another base station. The former method requires knowledge of the transmit timing, and thus, strict clock synchronization between the base stations and source. To eliminate the need for knowledge of the source transmit timing, differencing of arrival times at the receivers is commonly employed. Differencing the observed time of arrival eliminates some of the errors in TOA estimates common to all receivers and reduces other errors because of spatial and temporal coherence. While determining the TDOA from TOA estimates is a feasible method, cross-correlation techniques dominate the field of TDOA estimation techniques. As such, the discussion of TDOA estimation is limited to cross-correlation estimation techniques. In the following section, a general model for TDOA estimation is developed and the techniques for TDOA estimation are presented.

### 3.2.1 General Model for TDOA Estimation

For a signal, $s(t)$, radiating from a remote source through a channel with interference and noise, the general model for the time-delay estimation between received signals at two base stations, $x_1(t)$ and $x_2(t)$, is given by

$$\begin{aligned} x_1(t) &= A_1 s(t-d_1) + n_1(t) \\ x_2(t) &= A_2 s(t-d_2) + n_2(t), \end{aligned} \tag{3.1}$$

where $A_1$ and $A_2$ are the amplitude scaling of the signal, $n_1(t)$ and $n_2(t)$ consist of noise and interfering signals and $d_1$ and $d_2$ are the signal delay times, or arrival times. This model assumes that $s(t)$ , $n_1(t)$ and $n_2(t)$ are real and jointly stationary, zero-mean (time average) random processes and that $s(t)$ is uncorrelated with noise $n_1(t)$ and $n_2(t)$. Referring the delay time and scaling amplitudes to the receiver with the shortest time of arrival, assuming $d_1 < d_2$, the model of (3.1) can be rewritten as

$$\begin{aligned} x_1(t) &= s(t) + n_1(t) \\ x_2(t) &= As(t-D) + n_2(t), \end{aligned} \tag{3.2}$$

where $A$ is the amplitude ratio and $D = d_2 - d_1$. It is desired to estimate $D$, the time difference of arrival (TDOA) of $s(t)$ between the two receivers. It may also be desirable to estimate the scaling amplitude $A$. By estimating the amplitude scaling, selection of the appropriate receivers can be made. It follows that the limit cyclic cross-correlation and autocorrelations are given by

$$\begin{aligned} R^{\alpha}_{x_2x_1}(\tau) &= AR^{\alpha}_{s}(\tau - D)e^{-j\pi\alpha D} + R^{\alpha}_{n_2n_1}(\tau) && (3.3) \\ R^{\alpha}_{x_1}(\tau) &= R^{\alpha}_{s}(\tau) + R^{\alpha}_{n_1}(\tau) && (3.4) \\ R^{\alpha}_{x_2}(\tau) &= |A|^2R^{\alpha}_{s}(\tau)e^{-j\pi\alpha D} + R^{\alpha}_{n_2}(\tau), && (3.5) \end{aligned}$$

where the parameter $\alpha$ is called the cycle frequency [Gar92a]. If $\alpha = 0$, the above equations are the conventional limit cross-correlation and autocorrelations.

If $s(t)$ exhibits a cycle frequency $\alpha$ not shared by $n_1(t)$ and $n_2(t)$, then by using this values of $\alpha$ in the measurements in (3.3)-(3.5), we obtain through infinite time averaging

$$R^{\alpha}_{n_1}(\tau) = R^{\alpha}_{n_2}(\tau) = R^{\alpha}_{n_2n_1}(\tau) = 0 \qquad (3.6)$$

and the general model for time delay estimation between base stations is

$$\begin{aligned} R^{\alpha}_{x_2x_1}(\tau) &= AR^{\alpha}_{s}(\tau - D)e^{-j\pi\alpha D} && (3.7) \\ R^{\alpha}_{x_1}(\tau) &= R^{\alpha}_{s}(\tau) && (3.8) \\ R^{\alpha}_{x_2}(\tau) &= |A|^2R^{\alpha}_{s}(\tau)e^{-j\pi\alpha D}. && (3.9) \end{aligned}$$

Accurate TDOA estimation requires the use of time delay estimation techniques that provide resistance to noise and interference and the ability to resolve multipath signal components. Many techniques have been developed that estimate TDOA $D$ with varying degrees of accuracy and robustness. These include the generalized cross-correlation (GCC) and cyclostationarity-exploiting cross-correlation methods. Cyclostationarity-exploiting methods include the Cyclic Cross-Correlation (CYC-COR), the Spectral-Coherence Alignment (SPECCOA) method, the Band-Limited Spectral Correlation Ratio (BL-SPECCORR) method and the Cyclic Prony method [Gar94]. While signal selective cyclostationarity-exploiting methods have been shown in [Gar94] and [Gar92b] to outperform GCC methods in the presence of noise and interference, they do so only when spectrally overlapping noise and interference exhibit a cycle frequency different than the signal of interest. When spectrally overlapping signals exhibit the same cycle frequency, as is encountered in multiuser CDMA systems, these methods do not offer an advantage over GCC methods. As such, only generalized cross-correlation methods for TDOA estimation are presented.

### 3.2.2 Generalized Cross-Correlation Methods

Conventional correlation techniques that have been used to solve the problem of TDOA estimation are referred to as generalized cross-correlation (GCC) methods. These methods have been explored in [Gar92a], [Gar92b], [Kna76], [Car87], [Rot71], [Hah73] and [Hah75]. These GCC methods cross-correlate prefiltered versions of the received signals at two receiving stations, then estimate the TDOA $D$ between the two stations as the location of the peak of the cross-correlation estimate. Prefiltering is intended to accentuate frequencies for which high signal-to-noise (SNR) is highest and attenuate the noise power before the signal is passed to the correlator.

Generalized cross-correlation methods for TDOA estimation are based on (3.7) with $\alpha = 0$ [Gar92a]. Thus (3.7) is rewritten as

$$R^0_{x_2x_1}(\tau) = AR^0_s(\tau - D). \tag{3.10}$$

The argument $\tau$ that maximizes (3.10) provides an estimate of the TDOA $D$. Equivalently, (3.10) can be written as

$$R_{x_2x_1}(\tau) = R^0_{x_2x_1}(\tau) = \int_{-\infty}^{\infty} x_1(t)x_2(t-\tau)dt. \tag{3.11}$$

However, $R_{x_2x_1}(\tau)$ can only be estimated from a finite observation time. Thus, an estimate of the cross-correlation is given by

$$\hat{R}_{x_2x_1}(\tau) = \frac{1}{T}\int_0^T x_1(t)x_2(t-\tau)dt, \tag{3.12}$$

where $T$ represents the observation interval. Equation (3.12) is based on the use of an analog correlator. An integrate and dump correlation receiver of this form is one realization of a matched filter receiver [Zie85]. The correlation process can also be implemented digitally if sufficient sampling of the waveform is used. The output of a discrete correlation process using digital samples of the signal is given by

$$\hat{R}_{x_2x_1}(m) = \frac{1}{N}\sum_{n=0}^{N-|m|-1} x_1(n)x_2(n+m)dt. \tag{3.13}$$

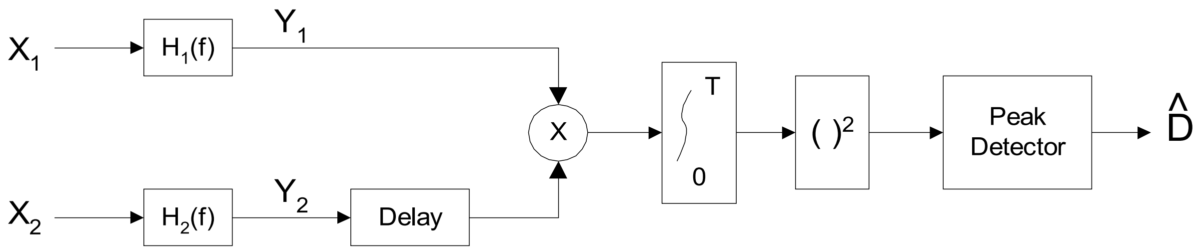


Figure 3.1: Generalized Cross-Correlation Method for TDOA Estimation

The cross-power spectral density function, $G_{x_2x_1}(f)$, related to the cross-correlation of $x_1(t)$ and $x_2(t)$ in (3.12) is given by

$$R_{x_2x_1}(\tau) = \int_{-\infty}^{\infty} G_{x_2x_1}(f)e^{j\pi f\tau}df \tag{3.14}$$

or

$$G_{x_2x_1}(f) = \int_{-\infty}^{\infty} R_{x_2x_1}(\tau)e^{-j\pi f\tau}dt. \tag{3.15}$$

As before, because only a finite observation time of $x_1(t)$ and $x_2(t)$ is possible, only an estimate $\hat{G}_{x_2x_1}(f)$ of $G_{x_2x_1}(f)$ can be obtained.

In order to improve the accuracy of the delay estimate, filtering of the two signals is performed before integrating in (3.12). As shown in Figure 3.1, each signal $x_1(t)$ and $x_2(t)$ is filtered through $H_1(f)$ and $H_2(f)$, then correlated, integrated and squared. This is performed for a range of time shifts, $\tau$, until a peak correlation is obtained. The time delay causing the cross-correlation peak is an estimate of the TDOA $\hat{D}$. If the correlator is to provide an unbiased estimate of TDOA $D$, the filters must exhibit the same phase characteristics and hence are usually taken to be identical filters [Hah73].

When $x_1(t)$ and $x_2(t)$ are filtered, the cross-power spectrum between the filtered outputs is given by

$$G_{y_2y_1}(f) = H_1(f)H_2^*(f)G_{x_2x_1}(f), \tag{3.16}$$

where $*$ denotes the complex conjugate. Therefore, the generalized cross-correlation, specified by superscript $G$, between $x_1(t)$ and $x_2(t)$ is

$$R_{y_2y_1}^G(\tau) = \int_{-\infty}^{\infty} \Psi_G(f)G_{x_2x_1}(f)e^{j\pi f\tau}df, \tag{3.17}$$

where

$$\Psi_G(f) = H_1(f)H_2^*(f) \tag{3.18}$$

and denotes the general frequency weighting, or filter function. Because only an estimate of $R^G_{y_2y_1}(\tau)$ can be obtained, (3.17) is rewritten as

$$\hat{R}^G_{y_2y_1}(\tau) = \int_{-\infty}^{\infty} \Psi_G(f)\hat{G}_{x_2x_1}(f)e^{j\pi f\tau}df, \tag{3.19}$$

which is used to estimate the TDOA $D$. The GCC methods use filter functions $\Psi_G(f)$ to minimize the effect of noise and interference.

The choice of the frequency function, $\Psi_G(f)$, is very important, especially when the signal has multiple delays resulting from a multipath environment. Consider the optimal case in which $n_1(t)$ and $n_2(t)$ are uncorrelated and only one signal delay is present. The cross-correlation of $x_1(t)$ and $x_2(t)$ in (3.10) can be rewritten as

$$R^0_{x_2x_1}(\tau) = AR^0_s(\tau) \otimes \delta(t - D), \tag{3.20}$$

where $\otimes$ denotes a convolution operation. Equation (3.20) can be interpreted as the spreading of a delta function at $D$ by the inverse Fourier transform of the signal spectrum. When the signal experiences multiple delays due to a multipath environment, the cross-correlation can be represented as

$$R_{x_2x_1}(0) = R^0_s(\tau) \otimes \sum_i A_i\delta(t - D_i). \tag{3.21}$$

If the delays of the signal are not sufficiently separated, the spreading of the one delta function will overlap another, thereby making the estimation of the peak and TDOA difficult if not impossible. The frequency function $\Psi_G(f)$ can be chosen to ensure a large peak in the cross-correlation $x_1(t)$ and $x_2(t)$, resulting in a narrower spectra and better TDOA resolution. However, in doing so, the peaks are more sensitive to errors introduced by the finite observation time, especially in cases of low signal to noise ratio (SNR). Thus the choice of $\Psi_G(f)$ is a compromise between good resolution and stability [Kna76].

Several frequency functions, or processors, have been proposed to facilitate the estimate of $\hat{D}$. When the filters $H_1(f) = H_2(f) = 1$, $\forall(f)$, then $\Psi_G(f) = 1$, and the estimate $\hat{G}$ is simply the delay abscissa at which the cross-correlation peaks. This is considered cross-correlation processing. Other processors include the Roth Impulse

Table 3.1: GCC Frequency Functions

| **Processor Name** | **Frequency Function** $\Psi_G(f)$ |
|---|---|
| Cross-correlation | 1 |
| Roth Impulse Response | $1/G_{x_1x_1}(f)$ or $1/G_{x_2x_2}(f)$ |
| Smoothed Coherence Transform | $1/\sqrt{G_{x_1x_1}(f)G_{x_2x_2}(f)}$ |
| Eckart | $G_{s_1s_1}(f)/[G_{n_1n_1}(f)G_{n_2n_2}(f)]$ |
| Hannon-Thomson or Maximum Likelihood | $\frac{\lvert\gamma_{x_1x_2}(f)\rvert^2}{\lvert G_{x_1x_2}(f)\rvert[1-\lvert\gamma_{x_1x_2}(f)\rvert^2]}$ |

Response processor [Rot71], the Smoothed Coherence Transform (SCOT) [Car73], the Eckart filter [Kna76],[Hah73], and the Hannan-Thomson (HT) processor or Maximum Likelihood (ML) estimator [Han73]. A list of GCC frequency functions is provided in Table 3.1

The Roth Impulse Response processor has the desirable effect of suppressing the frequency regions in which power spectral noise density, $G_{n_1n_1}(f)$ or $G_{n_2n_2}(f)$ , is large and the estimate of the cross power spectral signal density, $\hat{G}_{x_1x_2}(f)$ , is likely to be in error. However, the Roth processor does not minimize the spreading effect of the delta function whenever the power spectral noise density is not equal to some constant times the power spectral density of the signal, $G_{s_1s_1}(f)$ [Kna76]. Furthermore, one is uncertain as to whether the errors in $\hat{G}_{x_1x_2}(f)$ are due to frequency bands in which $G_{n_1n_1}(f)$ or $G_{n_2n_2}(f)$ large.

The uncertainty with the Roth processor led to the development of the proposed Smoothed Coherence Transform (SCOT). The SCOT processor suppresses frequency bands of high noise and assigns zero weight to bands where $G_{s_1s_1}(f) = 0$ . The SCOT frequency function is given as

$$\Psi_S(f) = 1/\sqrt{G_{x_1x_1}(f)G_{x_2x_2}(f)}. \tag{3.22}$$

This results in the cross-correlation

$$R^S_{y_2y_1}(\tau) = \int_{-\infty}^{\infty} \hat{\gamma}_{x_1x_2}(f)e^{j\pi f\tau}df, \tag{3.23}$$

where

$$\hat{\gamma}_{x_1x_2}(f)\hat{=}\frac{\hat{G}_{x_1x_2}(f)}{\sqrt{G_{x_1x_1}(f)G_{x_2x_2}(f)}} \tag{3.24}$$

is the coherence estimate [Kna76]. The SCOT processor assigns weighting according to signal-to-noise (SNR) characteristics. In terms of the noise characteristics, [Hah73] realizes the SCOT as

$$|\Psi_S(f)|^2 = 1/N. \tag{3.25}$$

The SCOT frequency function, for which $H_1(f) = 1/\sqrt{G_{x_1x_1}(f)}$ and $H_2(f) = 1/\sqrt{G_{x_2x_2}(f)}$, can be interpreted as a prewhitening process. If $G_{x_1x_1}(f) = G_{x_2x_2}(f)$, then the SCOT filter is equivalent to the Roth filter. Consequently, the SCOT still produces the same broadening as the Roth function [Kna76].

The Eckart processor, similarly to the SCOT processor, suppresses frequency bands of high noise and assigns zero weight to bands where $G_{s_1s_1}(f) = 0$ . The Eckart frequency function maximizes the ratio of the change in mean correlator output to the standard deviation of the correlator output due to the noise alone [Kna76]. For the model given by (3.2) and $n_1(t)$ and $n_2(t)$ having the same spectra, the Eckart frequency function in terms of the SNR characteristics is given by [Hah73] as

$$|\Psi_E(f)|^2 = S/N^2. \tag{3.26}$$

In practice, the Eckart filter requires knowledge of the signal and noise spectra [Kna76].

The previously described frequency functions have been shown to be suboptimal by [Hah73]. The HT processor, which is equivalent to a ML estimator, has been shown to be the optimal processor by [Hah73] and [Kna76]. The HT frequency function is given by

$$\Psi_{HT}(f) = \frac{|\gamma_{x_1x_2}(f)|^2}{|G_{x_1x_2}(f)|[1-|\gamma_{x_1x_2}(f)|^2]}, \tag{3.27}$$

where $|\gamma_{x_1x_2}(f)|^2$ is the magnitude-squared coherence [Car81]. For the model in (3.2) and $n_1(t)$ and $n_2(t)$ having the same spectra, the HT frequency function in terms of the SNR characteristics is given as

$$|\Psi_{HT}(f)|^2 = \frac{S/N^2}{1+2(S/N)}. \tag{3.28}$$

For low SNR, it has been shown in [Kna76] that the Eckart function is equivalent to the HT frequency function.

These GCC TDOA estimation methods have been shown to effective in reducing the effects of noise and interference [Gar92b]. However, if the noise and interference $n_1(t)$ and $n_2(t)$ in (3.2) are both temporally and spectrally coincident with $s(t)$, there is little that GCC methods can do to reduce the undesirable effects of this interference. In this situation, generalized cross-correlation methods encounter two problems. First, GCC methods experience a resolution problem. These GCC methods require the differences in the TDOAs for each signal to be greater than the widths of the cross-correlation functions so that the peaks can be resolved. Consequently, if the TDOAs are not sufficiently separated, the overlapping of cross correlations can introduce significant errors in the TDOA estimate. Second, if $s(t)$, $n_1(t)$ and $n_2(t)$ are resolvable, conventional GCC methods must still identify which of the multiple peaks is due to the signal of interest and interference. These problems arise because GCC methods are not signal selective and produce TDOA peaks for all signals in the received data unless they are spectrally disjoint and can be filtered out [Gar94].

### 3.2.3 Measures of TDOA Estimation Accuracy

The Cramér-Rao Lower Bound (CRLB) on the variance of an unbiased estimator is the standard benchmark against which conventional TDOA estimation methods are evaluated. The derivation of the CRLB is given in [Kna76]. The CRLB typically used is for evaluating stationary Gaussian signals in stationary Gaussian noise environments [Gar92b]. However, the BPSK PN signaling used in CDMA systems exhibit fundamental periodocities in the chip period, data period and PN code repetition period. The signal is therefore nonstationary (cyclostationary) and thus cannot be appropriately evaluated by the typical CRLB. Although the CRLB for nonstationary signals exist, it is very difficult to evaluate.

## 3.3 Hyperbolic Position Location Estimation

Accurate position location (PL) estimation of a source requires an efficient hyperbolic position location estimation algorithm. Once the TDOA information has been acquired, the hyperbolic PL algorithm will be responsible for producing an accurate and unambiguous solution to the position location problem. Many processing algorithms, with different complexity and restrictions, have been proposed for position location estimation based on TDOA estimates.

When base stations are placed in a linear fashion relative to the source, the estimation of the PL is simplified. Carter's beamforming method provides an exact solution for the source range and bearing [Car81]. However, it requires am extensive search over a set of possible source locations, which can become computationally intensive. Hahn's method estimates the source range and bearing from the weighted sum of ranges and bearings obtained from the TDOA's of every possible combination. This method is very sensitive to the choice of weights, which can be very complicated in obtaining, and is only valid for distant sources [Hah73]-[Hah75]. Abel and Smith provide an explicit solution that can achieve the Cramér-Rao Lower Bound (CRLB) in the small error region [Abe89].

When the base stations are placed arbitrarily relative to the source, which is typical scenario of a mobile unit within the infrastructure of a cellular/PCS system, the position fix becomes more complex. In this situation, the position location of a source is determined from the intersection of hyperbolic curves produced from the TDOA estimates. The set of equations that describe these hyperbolic curves are nonlinear and are not easily solved. If the set of nonlinear hyperbolic equations equals the number of unknown coordinates of the source, then the system is consistent and a unique solution can be determined from iterative techniques. For an inconsistent system, in which redundant range difference measurements are made, the problem of solving for the position location of the source becomes more difficult because no unique solution exists.

While direct nonlinear solutions to the inconsistent system can provide accurate results, they tend to be very computationally intensive. Consequently, linearization of these equations is commonly used to simplify the computation of the position location solution. One method of linearizing the equations is by Taylor-series expansion and retaining the first two terms. Other methods have also been used. For most situations, linearization of the nonlinear equations of ranging PL system does not introduce undue errors in the position location estimate. However, linearization can introduce significant errors when determining a PL solution in bad geometric dilution of precision (GDOP) situations. It has been shown by Bancroft [Ban85] that eliminating the second order terms can lead to significant errors in this situation. The effect of linearization of hyperbolic equations on the position location solution is also explored by Nicholson in [Nic73] and [Nic76].

For an inconsistent system of equations, some error criteria must be determined for selecting an optimum solution. Classical techniques for solving these equations include the Least Squares (LS) and Weighted Least Squares (WLS) methods. These techniques can achieve the Maximum Likelihood (ML) estimate which maximizes the probability that a particular position estimate is the true position location. If the range difference errors are uncorrelated and Gaussian distributed with zero mean and equal variances then the LS solution provides the ML estimate [Sta94]. If the variances are unequal then the WLS solution is the ML estimate. The WLS utilizes weighting coefficients inversely proportional to the variances of the range difference estimates. However, a problem exists because the variances are either not known a priori or difficult to estimate.

For arbitrarily placed base stations and a consistent system of equations, Fang provides an exact solution to the nonlinear equations [Fan90]. For arbitrarily distributed base stations and redundant TDOA estimates, the spherical-intersection (SX) [Sch87], spherical-interpolation (SI) [Fri87], [Smi87a], [Smi87b], [Abe87], Divide and Conquer (DAC) [Abe90], Chan's method [Cha94] and the Taylor Series [Foy76], [Tor84] methods can be used. The Taylor Series estimation method provides a more accurate solution, even at reasonable TDOA noise levels, than the other methods but is also more computationally intensive. Consequently, a tradeoff between position location accuracy and computational requirements exists.

### 3.3.1 General Model for Hyperbolic PL Estimation

A general model for the two dimensional (2-D) PL estimation of a source using $M$ base stations is developed. Referring all TDOA's to the first base station, which is assumed to be the base station controlling the call and first to receive the transmitted signal, let the index $i = 2, \ldots, M$, unless otherwise specified, $(x, y)$ be the source location and $(X_i, Y_i)$ be the known location of the ith receiver. The squared range distance between the source and the $i$th receiver is given as

$$\begin{aligned} R_i &= \sqrt{(X_i - x)^2 + (Y_i - y)^2} \\ &= \sqrt{X_i^2 + Y_i^2 - 2X_i x - 2Y_i y + x^2 + y^2}. \end{aligned} \tag{3.29}$$

The range difference between base stations with respect to the first arriving base station is

$$\begin{aligned} R_{i,1} &= c\, d_{i,1} = R_i - R_1 \\ &= \sqrt{(X_i - x)^2 + (Y_i - y)^2} - \sqrt{(X_1 - x)^2 + (Y_1 - y)^2}, \end{aligned} \tag{3.30}$$

where $c$ is the signal propagation speed, $R_{i,1}$ is the range difference distance between the first base station and the $ith$ base station, $R_1$ is the distance between the first base station and the source, and $d_{i,1}$ is the estimated TDOA between the first base station and $ith$ base station. This defines the set of nonlinear hyperbolic equations whose solution gives the 2-D coordinates of the source.

Solving the nonlinear equations of (3.30) is difficult. Consequently, linearizing this set of equations is commonly performed. One way of linearizing these equations is through the use of Taylor-series expansion and retaining the first two terms [Foy76] [Tor84]. An commonly used alternative method to the Taylor-series expansion method, presented in [Fri87], [Sch87], [Smi87a] and [Abe87], is to first transform the set of nonlinear equations in (3.30) into another set of equations. Rearranging the form of (3.30) into

$$R_i^2 = (R_{i,1} + R_1)^2, \tag{3.31}$$

equation (3.29) now can be rewritten as

$$R_{i,1}^2 + 2R_{i,1}R_1 + R_1^2 = X_i^2 + Y_i^2 - 2X_i x - 2Y_i y + x^2 + y^2. \tag{3.32}$$

Subtracting (3.29) at i = 1 from (3.32) results in

$$R_{i,1}^2 + 2R_{i,1}R_1 = X_i^2 + Y_i^2 - 2X_{i,1}x - 2Y_{i,1}y + x^2 + y^2, \tag{3.33}$$

where $X_{i,1}$ and $Y_{i,1}$ are equal to $X_i - X_1$ and $Y_i - Y_1$ respectively. The set of equations in (3.33) are now linear with the source location $(x, y)$ and the range of the first receiver to the source $R_1$ as the unknowns, and are more easily handled.

### 3.3.2 Hyperbolic Position Location Algorithms

For arbitrarily placed base stations and a consistent system of equations in which the number of equations equals the number of unknown source coordinates to be solved, Fang [Fan90] provides an exact solution to the equations of (3.33). For a 2-D hyperbolic PL system using three base stations to estimate the source location $(x, y)$, Fang establishes a coordinate system so that the first base station (BS) is located at $(0, 0)$, the second BS at $(x_2, 0)$ and the third BS at $(x_3, y_3)$. Realizing that for the first BS, where $i = 1$, $X_1 = Y_1 = 0$, and for the second BS, where $i = 2$, $Y_2 = 0$ , the following relationships are simplified

$$\begin{aligned} R_1 &= \sqrt{(X_1 - x)^2 + (Y_1 - y)^2} = \sqrt{x^2 + y^2} \\ X_{i,1} &= X_i - X_1 = X_i \\ Y_{i,1} &= Y_i - Y_1 = Y_i. \end{aligned}$$

Using these relationships, the equation of (3.33) can be rewritten as

$$\begin{aligned} 2R_{2,1}R_1 &= R_{2,1}^2 - X_i^2 + 2X_{i,1}x \\ 2R_{3,1}R_1 &= R_{3,1}^2 - (X_3^2 + Y_3^2) + 2X_{3,1}x + 2Y_{3,1}y. \end{aligned} \tag{3.34}$$

Equating the two equations (3.34) and simplifying results in

$$y = g * x + h, \tag{3.35}$$

where

$$\begin{aligned} g &= \{R_{3,1} - (X_2/R_{2,1}) - X_3\}/Y_3 \\ h &= \{X_3^2 + Y_3^2 - R_{3,1}^2 + R_{3,1} * R_{2,1}(1 - (X_2/R_{2,1})^2)\}/2Y_3. \end{aligned}$$

Substituting equation (3.35) into the first equation in (3.34) results in

$$d * x^2 + e * x + f = 0, \tag{3.36}$$

where

$$\begin{aligned} d &= -\{(1 - (X_2/R_{2,1})^2) + g\} \\ e &= X_2 * \{(1 - (X_2/R_{2,1})^2)\} - 2g * h \\ f &= -(R_{2,1}^2/4) * \{(1 - (X_2/R_{2,1})^2)\}^2 - h^2. \end{aligned}$$

Fang's method provides an exact solution, however, his solution does not make use of redundant measurements made at additional receivers to improve position location accuracy. Furthermore, his method experiences an ambiguity problem due to the inherent squaring operation. These ambiguities can be resolved using a priori information or through use of symmetry properties.

To obtain a precise position estimate at reasonable noise levels, the Taylor-series method [Foy76], [Tor84] can be employed. The Taylor-series method linearizes the set of equations in (3.30) by Taylor-series expansion then uses an iterative method to solve the system of linear equations. The iterative method begins with an initial guess and improves the estimate at each iteration by determining the local linear least-square (LS) solution. With a set of TDOA estimates, the method starts with an initial guess $(x_0, y_0)$ and computes the deviations of the position location estimation

$$\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = (\mathbf{G}_t^T \mathbf{Q}^{-1} \mathbf{G}_t)^{-1} \mathbf{G}_t^T \mathbf{Q}^{-1} \mathbf{h}_t, \tag{3.37}$$

where

$$\mathbf{h}_t = \begin{bmatrix} R_{2,1} - (R_2 - R_1) \\ R_{3,1} - (R_3 - R_1) \\ \vdots \\ R_{M,1} - (R_M - R_1) \end{bmatrix}$$

$$\mathbf{G}_t = \begin{bmatrix} [(X_1 - x)/R_1] - [(X_2 - x)/R_2] & [(Y_1 - y)/R_1] - [(Y_2 - y)/R_2] \\ [(X_1 - x)/R_1] - [(X_3 - x)/R_3] & [(Y_1 - y)/R_1] - [(Y_3 - y)/R_3] \\ \vdots & \vdots \\ [(X_1 - x)/R_1] - [(X_M - x)/R_M] & [(Y_1 - y)/R_1] - [(Y_M - y)/R_M] \end{bmatrix},$$

and $\mathbf{Q}$ is the covariance matrix of the estimated TDOA's. The values $R_i$ for $i = 1, 2, \ldots, M$ are computed from (3.29) with $x = x_0$ and $y = y_0$. In the next iteration, $x_0$ and $y_0$ are set to $x_0 + \Delta x$ and $y_0 + \Delta y$. The whole process is repeated until $\Delta x$ and $\Delta y$ are sufficiently small, resulting in the estimated PL of the source $(x, y)$. This Taylor-series method can provide accurate results, however, it requires a close initial guess $(x_0, y_0)$ to guarantee convergence and can be very computationally intensive.

Friedlander's method [Fri87] utilizes a Least Squares (LS) and Weighted LS (WLS) error criterion to solve for the position location [Fri87]. He first transforms the linear set of equations of (3.33) into

$$X_{i,1}x + Y_{i,1}y = \frac{1}{2}(X_i^2 + Y_i^2 - X_1^2 - Y_1^2 - R_{i,1}^2) - R_{i,1}R_1, \tag{3.38}$$

then realizes this equation in matrix form as

$$\mathbf{Sx} = \mathbf{u} - \mathbf{R}_1\mathbf{p}, \tag{3.39}$$

where

$$\mathbf{S} = \begin{bmatrix} X_{i,1} & Y_{i,1} \\ \vdots & \vdots \\ X_{M,1} & Y_{M,1} \end{bmatrix}$$

$$\mathbf{x} = [x \quad y]^T$$

$$\mathbf{u} = \frac{1}{2}\begin{bmatrix} X_i^2 + Y_i^2 - X_1^2 - Y_1^2 - R_{i,1}^2 & \\ \vdots & \vdots \\ X_M^2 + Y_M^2 - X_1^2 - Y_1^2 - R_{M,1}^2 & \end{bmatrix}$$

$$\mathbf{p} = [\mathbf{R}_{i,1} \ldots \mathbf{R}_{M,1}]^T.$$

In order to eliminate the second term of 3.39, which requires knowledge of the unknown term $R_1$, the equation in (3.39) is premultiplied by a matrix $\mathbf{N}$ which has $\mathbf{p}$ in its null-space. Matrix $\mathbf{N}$ is defined as

$$\mathbf{N} = (\mathbf{I} - \mathbf{Z})\mathbf{D}, \tag{3.40}$$

where

$$\mathbf{D} = (diag\{\mathbf{p}\})^{-1} = \begin{bmatrix} R_{i,1} & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & R_{M,1} \end{bmatrix}^{-1}$$

$$\mathbf{Z} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{bmatrix},$$

and $\mathbf{I}$ is an identity matrix. By using singular value decomposition (SVD) on $(\mathbf{I} - \mathbf{Z})$, Friedlander eliminates the unknown $R_1$ parameter. The SVD of $(\mathbf{I} - \mathbf{Z})$ is given as

$$(\mathbf{I} - \mathbf{Z}) = [\mathbf{U}_k, \mathbf{u}_k] \begin{bmatrix} \eta_1^k & & & 0 \\ & \ddots & & \\ & & \eta_{M-2}^k & \\ 0 & & & 0 \end{bmatrix} [\mathbf{V}_k, \mathbf{v}_k^T], \tag{3.41}$$

where $\{\eta_1^k, \ldots, \eta_{M-2}^k\}$ are the non-zero singular values. The matrix to eliminate the second term of (3.39) is then given by

$$\mathbf{N} = \mathbf{V}_k^T \mathbf{D}, \tag{3.42}$$

and a closed form solution for the coordinates of the source is found by solving

$$\mathbf{NSx} = \mathbf{Nu}, \tag{3.43}$$

which results in $(M - 2)$ linear equations in $x$ and $y$. The source position can then be computed using the LS solution. A closed form solution which can be used is given by Friedlander as

$$\mathbf{x} = (\mathbf{S}^T \mathbf{N} \mathbf{N}^T \mathbf{S})^{-1} \mathbf{S}^T \mathbf{N} \mathbf{N}^T \mathbf{u}. \tag{3.44}$$

Friedlander also uses the Weighted Least Squares (WLS) solution, which provides the ML estimate in the case of zero-mean Gaussian range difference noises with unequal variance. The optimal weighting matrix is given as

$$\mathbf{W} = \{\mathbf{N}(diag\{\mathbf{p}\} + R_1 \mathbf{I}) \mathbf{Q} (diag\{\mathbf{p}\} + R_1 \mathbf{I}) \mathbf{N}^T\}^{-1}, \tag{3.45}$$

where $\mathbf{Q}$ is the covariance matrix of the range difference equations. The weighting matrix $\mathbf{W}$ is a function of the unknown parameter $R_1$ , which can be estimated from

$$R_1 = \frac{\mathbf{p}^T \mathbf{P} \mathbf{u}}{\mathbf{p}^T \mathbf{P} \mathbf{p}}, \tag{3.46}$$

where

$$\mathbf{P} = \mathbf{I} - \mathbf{S}(\mathbf{S}^T \mathbf{S})^{-1} \mathbf{S}.$$

The resulting WLS solution is in the form

$$\mathbf{W}^{1/2}\mathbf{N}\mathbf{x} = \mathbf{W}^{1/2}\mathbf{N}\mathbf{u}, \tag{3.47}$$

which results in a position location estimation by

$$\mathbf{x} = (\mathbf{S}^T\mathbf{N}\mathbf{W}\mathbf{N}^T\mathbf{S})^{-1}\mathbf{S}^T\mathbf{N}\mathbf{W}\mathbf{N}^T\mathbf{u}. \tag{3.48}$$

Friedlander's simulation results indicated that, when using four base stations, the LS and WLS solutions were identical. However, for more than four base stations, the WLS PL solution outperformed the LS PL solution.

The spherical-intersection (SX) method [Sch87], [Smi87b] is another commonly used approach. It assumes that $R_1$ is known and solves $x$ and $y$ in terms of $R_1$ from (3.33). The least squares solution of (3.29) is then used to find the $R_1$ and hence $x$ and $y$. Since $R_1$ is assumed to be constant in the first step, the degree of freedom to minimize the second norm of the error vector, $\psi$ , used in the solution is reduced [Cha94]. The solution obtained is therefore suboptimal as demonstrated in [Smi87a] and [Abe87].

Another approach called the spherical-interpolation (SI) method [Abe87], [Smi87a], [Smi87b], first solves $x$ and $y$ in terms of $R_1$, then inserts the intermediate result back into (3.33) to generate equations in the unknown $R_1$ only. Substituting the computed $R_1$ values that minimizes the LS equation error to the intermediate result produces the final result. One drawback to the SI method is it inability to produce a solution if the number of unknowns is equal to the number of equations based on the TDOA estimates, which may occur in certain situations. The SI method was shown in [Smi87a] to provide an order of magnitude greater noise immunity than the SX method. Although the SI performs better than the SX method, it assumes that the three variables $x$, $y$ and $R_1$ in (3.33) to be independent and eliminates $R_1$ from those equations. Consequently, the solution is suboptimal because this relationship is ignored. The method proposed by Friedlander and the SI method have be shown in [Fri87] to be mathematically equivalent.

A divide and conquer (DAC) method, proposed by Abel [Abe90], consists of dividing the TDOA measurements into groups, each having a size equal to the number of unknowns. Solution of the unknowns is calculated for each group, then appropriately combined to provide a final solution. Although this method can achieve optimum performance, the solution uses stochastic approximation and requires that the Fisher information be sufficiently large. The Fisher information matrix (FIM) is the inverse of the Cramér-Rao Matrix Bound (CRMB)(i.e.(FIM)=$(CRMB)^{-1}$) [Hah75]. The estimator provides optimum performance when the errors are small, thus implying a low-noise threshold in which the method deviates from the CRLB. This method requires an equal number of range difference measurements in each group, and as a result, the TDOA estimates from the remaining sensors cannot be used to improve accuracy.

An non-iterative solution to the hyperbolic position estimation problem which is capable of achieving optimum performance for arbitrarily placed sensors was proposed by Chan [Cha94]. The solution is in closed-form and valid for both distant and close sources. When TDOA estimation errors are small, this method is an approximation to the maximum likelihood (ML) estimator. Chan's method performs significantly better than the SI method and has a higher noise threshold that the DAC method before the performance deviates from the Cramér-Rao lower bound. Furthermore, it provides an explicit solution form that is not available in the Taylor-series method.

Following Chan's method [Cha94], for a three base station system (M=3), producing two TDOA's, $x$ and $y$ can be solved in terms of $R_1$ from (3.33). The solution is in the form of

$$\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = - \begin{bmatrix} X_{2,1} & Y_{2,1} \\ X_{3,1} & Y_{3,1} \end{bmatrix}^{-1} \times \left\{ \begin{bmatrix} R_{2,1} \\ R_{3,1} \end{bmatrix} R_1 + \frac{1}{2} \begin{bmatrix} R_{2,1}^2 - K_2 + K_1 \\ R_{3,1}^2 - K_3 + K_1 \end{bmatrix}, \right\} \tag{3.49}$$

where

$$\begin{aligned} K_1 &= X_1^2 + Y_1^2 \\ K_2 &= X_2^2 + Y_2^2 \\ K_3 &= X_3^2 + Y_3^2. \end{aligned}$$

When (3.49) is inserted into (3.29), with $i = 1$, a quadratic equation in terms of $R_1$ is produced. Substituting the positive root back into (3.49) results in the final solution. There may exist two positive roots from the quadratic equation that can produce two different solutions, resulting in an ambiguity. This problem can be resolved by using a *priori* information. The answer to this method is the same as the spherical-intersection method (SX) given in [Sch87].

For four or more base stations, the PL system is overdetermined because there are more measurements of TDOA than the number of unknowns. The original set of nonlinear TDOA equations are transformed into another set of linear equations with an extra variable. A weighted linear LS provides an initial solution and a second weighted LS gives an improved position estimate through the use of the known constrain of the source coordinates and the extra variable. Letting $\mathbf{z}_a = [\mathbf{z}_p^T, R_1]^T$ be the unknown vector, where $\mathbf{z}_p = [x, y]^T$, the error vector $\psi$, with TDOA noise, derived from (3.33) is

$$\psi = \mathbf{h} - \mathbf{G}_a \mathbf{z}_a^0, \tag{3.50}$$

where

$$\mathbf{h} = \frac{1}{2} \begin{bmatrix} R_{2,1}^2 - X_2^2 - Y_2^2 + X_1^2 + Y_1^2 \\ R_{3,1}^2 - X_3^2 - Y_3^2 + X_1^2 + Y_1^2 \\ \vdots \\ R_{M,1}^2 - X_M^2 - Y_M^2 + X_1^2 + Y_1^2 \end{bmatrix}$$

$$\mathbf{G}_a = - \begin{bmatrix} X_{2,1} & Y_{2,1} & R_{2,1} \\ X_{3,1} & Y_{3,1} & R_{3,1} \\ \vdots & \vdots & \vdots \\ X_{M,1} & Y_{M,1} & R_{M,1} \end{bmatrix}.$$

Denoting the noise free value of $\{*\}$ as $\{*\}^0$, when $d_{i,j} = d_{i,j}^0 + n_{i,j}$ is used to express $R_{i,1}$ as $R_{i,1}^0 + cn_{i,1}$ and noting from (3.30) that $R_i^0 = R_{i,1}^0 + R_i^0$, the error vector $\psi$ is found to be

$$\psi = c\mathbf{B}\mathbf{n} + 0.5c^2 \mathbf{n} \odot \mathbf{n} \tag{3.51}$$

$$\mathbf{B} = diag\{R_2^0, R_3^0, \ldots, R_M^0\},$$

where $\odot$ represents the Schur product. The TDOA estimates found by the general cross-correlation methods with Gaussian data is asymptotically normally distributed when the signal to noise (SNR) is high [Car81]. Consequently, the noise vector $\mathbf{n}$ is also asymptotically normal and the covariance matrix of the error vector can be evaluated. In practice, the condition $cn_{i,1} \ll R_i^0$ is usually satisfied and the second term on the right of (3.51) can be ignored. Therefore, the error vector $\psi$ becomes a Gaussian random vector with covariance matrix given by

$$\mathbf{\Psi} = \mathbf{E}[\psi\psi^T] = c^2 \mathbf{BQB} \tag{3.52}$$

where $\mathbf{Q}$ is the TDOA covariance matrix. The elements of $\mathbf{z}_a$ are related by (3.29), which means that (3.50) is still a set of nonlinear equations in $x$ and $y$ variables.

The approach to solving this nonlinear problem is to first assume that there is no relationship between $x, y$ and $R_1$. They can then be solved by a least squares estimation. The final solution is obtained by imposing the known relationship (3.29) to the result via another LS computation. This two step procedure is an approximation of a true ML estimator for the source location. By considering the elements of $\mathbf{z}_a$ independent, the ML estimate of $\mathbf{z}_a$ is

$$\begin{aligned}\mathbf{z}_a &= arg\,min\{(\mathbf{h}-\mathbf{G}_a\mathbf{z}_a)^T\mathbf{\Psi}^{-1}(\mathbf{h}-\mathbf{G}_a\mathbf{z}_a)\} \\ &= (\mathbf{G}_a^T\Psi^{-1}\mathbf{G}_a)^{-1}\mathbf{G}_a^T\mathbf{\Psi}^{-1}\mathbf{h},\end{aligned} \tag{3.53}$$

which is the generalized LS solution of (3.50). The equation of (3.53) cannot be solved because $\mathbf{\Psi}$ is not known since $\mathbf{B}$ contains the true distances between the source and the sensors. Further approximation is necessary in order to make the problem solvable.

When the source is far away, each $R_i^0$ for i=2,3,4,...,M is close to $R^0$ so that $\mathbf{B} \approx R^0\mathbf{I}$, where $R^0$ designates the range and $\mathbf{I}$ is the identity matrix size (M-1). Since scaling of $\mathbf{\Psi}$ does not affect the answer, an approximation of (3.53) is

$$\mathbf{z}_a \approx (\mathbf{G}_a^T\mathbf{Q}^{-1}\mathbf{G}_a)^{-1}\mathbf{G}_a^T\mathbf{Q}^{-1}\mathbf{h}. \tag{3.54}$$

If the source is close, (3.54) is used first to obtain an initial solution to estimate $\mathbf{B}$. The final answer is then computed from (3.53). Although (3.53) can be iterated to provide an even better answer, simulations show that applying (3.53) once is sufficient to give an accurate answer.

The covariance of position estimate $\mathbf{z}_a$ is obtained by evaluating the expectations of $\mathbf{z}_a$ and $\mathbf{z}_a\mathbf{z}_a^T$ from (3.53), however this is difficult because $\mathbf{G}_a$ contains random quantities $r_{i,1}$. The covariance matrix is computed by using a perturbation approach. In the presence of noise

$$R_{i,1} = R_{i,1}^0 + c\, n_{i,1}. \tag{3.55}$$

The matrix $\mathbf{G}_a$ and vector $\mathbf{h}$ can be expressed as $\mathbf{G}_a = \mathbf{G}_a^0 + \Delta\mathbf{G}_a$ and $\mathbf{h} = h^0 + \Delta\mathbf{h}$. Since $\mathbf{G}_a^0\mathbf{z}_a^0 = \mathbf{h}^0$, (3.50) implies that

$$\psi = \mathbf{\Delta h} - \mathbf{\Delta G}_a\mathbf{z}_a^0. \tag{3.56}$$

Letting $\mathbf{z}_a = \mathbf{z}_a^0 + \Delta\mathbf{z}_a$. Then from (3.53)

$$(\mathbf{G}_a^{0T} + \Delta\mathbf{G}_a^T)\mathbf{\Psi}^{-1}(\mathbf{G}_a^0 + \Delta\mathbf{G}_a)(\mathbf{z}_a^0 + \Delta\mathbf{z}_a) = (\mathbf{G}_a^{0T} + \Delta\mathbf{G}_a^T)\mathbf{\Psi}^{-1}(\mathbf{h} + \Delta\mathbf{h}). \quad (3.57)$$

Retaining only the linear perturbation terms and then using (3.51) and (3.56), $\mathbf{\Delta z}_a$ and its covariance matrix is

$$\begin{aligned} \Delta\mathbf{z}_a &= c(\mathbf{G}_a^T\mathbf{\Psi}^{-1}\mathbf{G}_a)^{-1}\mathbf{G}_a^T\mathbf{\Psi}^{-1}\mathbf{Bn} \\ cov(\mathbf{z}_a) &= \mathbf{E}[\Delta\mathbf{z}_a\Delta\mathbf{z}_a^T] = (\mathbf{G}_a^{0T}\mathbf{\Psi}^{-1}\mathbf{G}_a^0)^{-1}, \end{aligned} \quad (3.58)$$

where the square error term in (3.51) has been ignored and (3.52) has been used to give $cov(\mathbf{z}_a)$.

The position estimation $\mathbf{z}_a$ assumes $x, y$ and $R_i$ are independent; however, they are related by (3.29). Therefore, we can incorporate this relationship to get an improved estimate. When the bias is ignored because of small errors in the TDOA estimates, the vector $\mathbf{z}_a$ is a random vector with its mean centered at the true value with covariance matrix (3.58). Hence the elements of $\mathbf{z}_a$ can be expressed as

$$z_{a,1} = x^0 + e_1, \quad z_{a,2} = y^0 + e_2, \quad z_{a,3} = R_1^0 + e_1, \quad (3.59)$$

where $e_1, e_2$ and $e_3$ are the estimation errors of $\mathbf{z}_a$ . Subtracting the first two elements of $\mathbf{z}_a$ by $X_1$ and $Y_1$, and then squaring the elements gives another set of equations

$$\psi' = \mathbf{h}' - \mathbf{G}_a'\mathbf{z}_a'^0, \quad (3.60)$$

where $\psi'$ is a vector denoting the inaccuracies in $\mathbf{z}_a$. Substituting (3.59) into (3.60) results in

$$\begin{aligned} \Psi_1' &= 2(x^0 - X_1)e_1 + e_1^2 \approx 2(x^0 - X_1)e_1 \\ \Psi_2' &= 2(y^0 - Y_1)e_2 + e_2^2 \approx 2(y^0 - Y_1)e_2 \\ \Psi_3' &= 2R_1^0e_3 + e_3^2 \approx 2R_1^0e_3. \end{aligned} \quad (3.61)$$

The approximation to the ML procedure used here is valid only if the errors $e_i$ are small. The covariance matrix of $\psi'$ is given by

$$\begin{aligned} \mathbf{\Psi}' &= [\psi'\psi'^T]4\mathbf{B}'cov(\mathbf{z}_a)\mathbf{B}' \\ \mathbf{B}' &= diag\{x^0 - X_1, y^0 - Y_1, R_1^0\}. \end{aligned} \quad (3.62)$$

Since $\psi$ is Gaussian, then it follows that $\psi'$ is also Gaussian, thus the ML estimate of $\mathbf{z}_a'$ is

$$\mathbf{z}_a' = (\mathbf{G}_a'^T \mathbf{\Psi}'^{-1} \mathbf{G}_a')^{-1} \mathbf{G}_a'^T \mathbf{\Psi}'^{-1} \mathbf{h}'. \tag{3.63}$$

The matrix $\mathbf{\Psi}'$is not known since it contains the true values of the position. Neverthe-less, $\mathbf{B}'$ can be approximated by using the values $\mathbf{z}_a$ and $\mathbf{G}_a^0$ in (3.58) approximated by $\mathbf{G}_a$ and $\mathbf{B}$ in (3.52) approximated by the values computed from (3.54).

If the source is distant, then the covariance matrix can be approximated as

$$cov(\mathbf{z}_a) \approx c^2 R^{0^2} (\mathbf{G}_a^{0T} \mathbf{Q}^{-1} \mathbf{G}_a^0)^{-1}, \tag{3.64}$$

and (3.63) reduces to

$$\mathbf{z}_a' \approx (\mathbf{G}_a'^T \mathbf{B}'^{-1} \mathbf{G}_a \mathbf{Q}^{-1} \mathbf{G}_a \mathbf{B}'^{-1} \mathbf{G}_a')^{-1} (\mathbf{G}_a'^T \mathbf{B}'^{-1} \mathbf{G}_a \mathbf{Q}^{-1} \mathbf{G}_a \mathbf{B}'^{-1} \mathbf{G}_a') \mathbf{h}'. \tag{3.65}$$

Matrix $\mathbf{G}_a'$ is constant. By taking the expectations of $\mathbf{z}_a'$ and $\mathbf{z}_a' \mathbf{z}_a'^T$, the covariance matrix of $\mathbf{z}_a'$ is

$$cov(\mathbf{z}_a') = (\mathbf{G}_a'^T \mathbf{\Psi}'^{-1} \mathbf{G}_a')^{-1}. \tag{3.66}$$

Finally, the position location estimation is obtained from as

$$\mathbf{z}_p = \sqrt{\mathbf{z}_a'} + \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} \tag{3.67}$$

or

$$\mathbf{z}_p = -\sqrt{\mathbf{z}_a'} + \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix}.$$

The proper solution is selected to be the one which lies in the region of interest. If one of the coordinates of $\mathbf{z}_a'$ is close to zero, the square root in (3.67) may become imaginary. If this occurs, the imaginary component is set to zero. The covariance matrix of the position location estimate can be determined from $\mathbf{z}_a'$ in (3.60), with $x = x^0 + e_x$ and $y = y^0 + e_y$. Thus from (3.60)

$$\begin{aligned} z_{a,1}' - (x^0 - X_1)^2 &= 2(x^0 - X_1)e_x + e_x^2 \\ z_{a,2}' - (y^0 - Y_1)^2 &= 2(y^0 - Y_1)e_y + e_y^2. \end{aligned} \tag{3.68}$$

The errors $e_x$ and $e_y$ are relatively small compared to $x^0$ and $y^0$, thus we can eliminate $e_x^2$ and $e_y^2$. Then using (3.52), (3.58) , (3.62) and (3.67), the covariance matrix, $\mathbf{\Phi}$, of position location estimate, $\mathbf{z}_p$,is found to be

$$\begin{aligned}\mathbf{\Phi} &= cov(\mathbf{z}_p) = \frac{1}{4}\mathbf{B}''^{-1}cov(\mathbf{z}_a')\mathbf{B}''^{-1} \\ &= c^2(\mathbf{B}''\mathbf{G}_a'^T\mathbf{B}'^{-1}\mathbf{G}_a^{0T}\mathbf{B}^{-1}\mathbf{Q}^{-1}\mathbf{B}^{-1}\mathbf{G}_a^0\mathbf{B}'^{-1}\mathbf{G}_a'\mathbf{B}'')^{-1},\end{aligned} \tag{3.69}$$

where

$$\mathbf{B}'' = \begin{bmatrix} (x^0 - X_1) & 0 \\ 0 & ((y^0 - Y_1) \end{bmatrix}.$$

To summarize Chan's method, the location of a distant source can be estimated by using equations (3.54), (3.65) and (3.67). The covariance matrix of the PL estimate is determined from (3.69). For locating near sources, (3.54) is used to give an approximation of $\mathbf{B}$, then used in (3.53), (3.63) and (3.67) to produce a PL solution. The covariance matrix of the PL estimates can then be determined from (3.69). Chan's method offers a closed form solution which can achieve the CRLB. However, it does so when the range difference errors are assumed to be small. His method also requires a *priori* knowledge of the approximate location and distance of the source to resolve ambiguities in the PL solution.

The hyperbolic PL estimation algorithms presented offer different accuracy's and complexities. The Taylor-series LS method offers accurate position location estimation at reasonable noise levels and is applicable to any number of range difference measurements, but can be computational intensive. Fang's method provides an optimal solution when the system of equations is consistent but does not make use of redundant measurements. Friedlander's approach reduces the computational requirements for the solution but does is suboptimal because it eliminates a fundamental relationship. Chan's method offers a closed form solution, thus eliminating the need for an iteration approach, but requires *a priori* information to eliminate ambiguities. The optimal PL algorithm for a given situation depends on the geometrical configuration of the base stations, the number of coordinates of the source to be solved and range difference measurements utilized, computational requirements and complexity, assumptions on the statistical nature of the channel and desired accuracy.

## 3.4 Measures of Position Location Accuracy

A set of benchmarks is required to evaluate the accuracy of the hyperbolic position location technique. A commonly used measure of PL accuracy is the comparison of the mean square error (MSE) of the position location solution to the theoretical MSE based on the Cramér-Rao Lower Bound (CRLB). Another commonly used measure of PL accuracy is the circle of error probability (CEP). The effect of the geometric configuration of the base stations on the accuracy of the position location estimate is measured by the geometric dilution of precision (GDOP). A simple relationship exists between the GDOP and CEP measures. Lee in [Lee75a] and [Lee75b] provides a novel procedure for assessing the accuracy of hyperbolic multilateration PL systems and limitations of the accuracy's. Hepsaydir and Yates provide a performance analysis of position locationing systems using existing CDMA networks in [Hep94].

### 3.4.1 MSE and the Cramér-Rao Lower Bound

A commonly used measure of accuracy of a PL estimator is the comparison of the mean squared error (MSE) of the PL solution to the theoretical MSE based on the Cramér-Rao Lower Bound on the variance of unbiased estimators. The classical method for computing the MSE of a 2-D position location estimate is

$$MSE = \varepsilon = \mathbf{E}[(x-\hat{x})^2 + (y-\hat{y})^2], \tag{3.70}$$

where $(x, y)$ is the coordinates of the source and $(\hat{x}, \hat{y})$ is the estimated position of the source. The root-mean square (RMS) position location error, which can also be used as a measure of PL accuracy, is calculated as the square root of the MSE

$$RMS = \sqrt{\varepsilon} = \sqrt{\mathbf{E}[(x-\hat{x})^2 + (y-\hat{y})^2]}. \tag{3.71}$$

To gauge the accuracy of the PL estimator, the calculated MSE or RMS PL is compared to the theoretical MSE based on the Cramér-Rao Lower Bound (CRLB). The conventional CRLB sets a lower bound for the variance of any unbiased parameter estimator and is typically used for a stationary Gaussian signal in the presence of stationary Gaussian noise [Gar92b]. For non-Gaussian and nonstationary (cyclostationary) signals and noise, alternate methods have been used to evaluate the performance of the estimators [Gar92b]. The derivation of the CRLB for Gaussian noise is provided in [Cha94], [Kna76], [Hah73] and [Hah75]. The CRLB on the PL covariance is given by Chan [Cha94] as

$$\mathbf{\Phi} = c^2(\mathbf{G}_t^T \mathbf{Q}^{-1} \mathbf{G}_t)^{-1}, \tag{3.72}$$

where $\mathbf{G}_t$ is defined in (3.37) with $(x, y, R_i) = (x^0, y^0, R_i^0)$, which are the actual coordinates of the source and the range of the first base station to the source, and matrix $\mathbf{Q}$ is the TDOA covariance matrix. The sum of the diagonal elements of $\mathbf{\Phi}$ defines the theoretical lower bound on the MSE of the PL estimator. Matrix $\mathbf{Q}$ may not be known in practice; however, if the noise power spectral densities are similar at the receivers, it can be replaced be a theoretical TDOA covariance matrix with diagonal elements of $\sigma_d^2$ and $0.5\sigma_d^2$ for all other elements, where $\sigma_d^2$ is the variance of the TDOA estimate [Cha94].

### 3.4.2 Circular Error Probability

A crude but simple measure of accuracy of position location estimates that is commonly used is the circular error probability (CEP) [Tor84] [Foy76]. The CEP is a measure of the uncertainty in the location estimator relative to its mean. For a 2-D system, the CEP is defined as the radius of a circle which contains half of the realizations of the random vector with the mean as its center. If the position location estimator is unbiased, the CEP is a measure of the uncertainty relative to the true transmitter position. If the estimator is biased and bound by bias $B$, then with a probability of one-half, a particular estimate is within a distance $B$ + CEP from the true transmitter position. Figure 3.2 illustrates the 2-D geometrical relations.

The CEP is a complicated function and is usually approximated. Details of its computation can be found in [Foy76] and [Tor84]. For hyperbolic position location estimator the CEP is approximated with an accuracy within approximately 10 % as

$$CEP \approx 0.75\sqrt{\sigma_x^2 + \sigma_y^2}, \tag{3.73}$$

where $\sigma_x^2$ and $\sigma_y^2$ are the variances in the estimated position [Tor84].

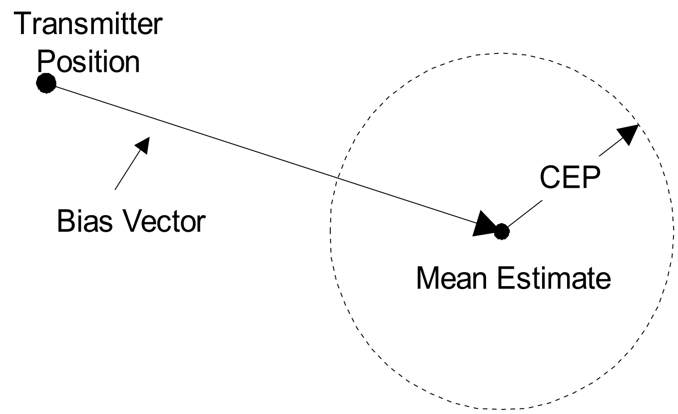


Figure 3.2: Circle of Error Probability

### 3.4.3 Geometric Dilution of Precision

The accuracy of Range-Based PL systems depends to a large extent on the geometric relationship between the base stations and the source to be located. One measure that quantifies the accuracy based on this geometric configuration is called the geometric dilution of precision (GDOP) [Wel86] [Tor84] [Ban85] [Lee75a]. The GDOP is defined as the ratio of the RMS position error to the RMS ranging error. The GDOP for an unbiased estimator and a ranging system is given by [Jor84] as

$$GDOP = \sqrt{tr[(\mathbf{A}^T\mathbf{A})^{-1}]}, \tag{3.74}$$

where $\mathbf{A}$ is express in equation (3.20) and $tr$ indicates the trace of the resulting matrix. The GDOP for an unbiased estimator and a 2-D hyperbolic system is given by [Tor84] and [Lee75a] as

$$\begin{aligned} GDOP &= \left(\sqrt{(c\sigma_x)^2 + (c\sigma_y)^2}\right)\Big/\sqrt{(c\sigma_s)^2} \\ &= \left(\sqrt{\sigma_x^2 + \sigma_y^2}\right)\Big/\sigma_s, \end{aligned} \tag{3.75}$$

where $(c\sigma_s)^2$ is the mean squared ranging error and $(c\sigma_x)^2$ and $(c\sigma_y)^2$ are the mean square position errors in the x and y estimates. The GDOP is related to the CEP by

$$CEP \approx (0.75\sigma_s)GDOP. \tag{3.76}$$

The GDOP can be used as a criterion for selecting a set of base station from a large set whose measurements produce minimum PL estimation error or for designing base station location within new systems.

## 3.5 Chapter Summary

This chapter introduced the TDOA estimation techniques and hyperbolic PL algorithms used in the hyperbolic position location method. A general model for the TDOA estimation problem was developed and the generalized cross-correlation (GCC) techniques commonly used for time delay estimation were presented. The effect of the frequency functions on the TDOA estimation and the importance of the choice of frequency function was discussed. Although GCC methods do facilitate the estimation of the TDOA, they do encounter problems which are critical to the position location problem. Firstly, GCC require the differences in the TDOA for each signal to be greater than the widths of the cross-correlation functions so that the correlation peaks can be resolved. If not separated sufficiently, overlapping of cross-correlation functions corrupt the TDOA estimate. Furthermore, because GCC methods are not signal-selective, they produce correlation peaks for all signals and are faced with the problem of identifying the TDOA estimate of interest. Signal-selective TDOA estimation techniques outperform GCC methods; however, they do not offer any advantages over GCC methods when the spectrally overlapping noise and interference exhibit the same cycle frequency as the signal of interest, which is encountered in multiuser CDMA systems.

A general range difference position location model was formulated and the hyperbolic algorithms used to provide solutions to the range difference equations were presented. For the problem of geolocating mobile units within a cellular infrastructure, the algorithms applicable to arbitrarily placed base station were reviewed. Fang's PL method, Friedlanders Least Square (LS) and Weighted LS PL method, the iterative Taylor-series LS PL method and Chan's closed form PL method were described in detail. The advantages and disadvantages of the various PL algorithms were discussed. Finally, the measures of position location accuracy commonly used to evaluate hyperbolic position location algorithms were reviewed.